MEM6810 Engineering Systems Modeling and Simulation
工程系统建模与仿真

Theory Analysis

Lecture 4: Random Variate Generation

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上海交通大学 SHANGHAI JIAO TONG UNIVERSITY | 董浩云智能制造与服务管理研究院 CY TUNG Institute of Intelligent Manufacturing and Service Management (中美物流研究院) (Sino-US Global Logistics Institute)

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Introduction

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    - E.g., 5 random variates (outcomes) from a $\mathcal{N}(0,1)$ random variable: $0.5377, 1.8339, -2.2588, 0.8622, 0.3188$.

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- When simulating a system, we often need to generate random variates (e.g., interarrival time, service time) from all kinds of distributions (e.g., exponential distribution, arbitrary empirical distribution).

Introduction

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- To produce a sequence of random variates from a given distribution (of a random variable):
  1. Start with random variates from $\mathrm{Unif}(0,1)$ (called **random numbers**).
  2. All random variates with given distribution are "transformed" from **random numbers**.

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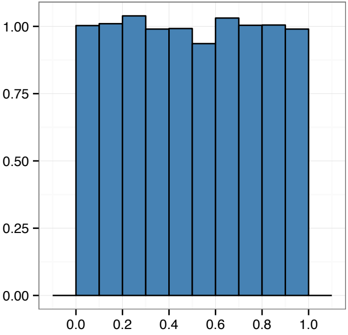
- Statistical Properties
  - Uniformity: Each value on $[0, 1]$ has equal likelihood.
  - Independence: Implies no correlation between successive numbers.

- Uniformity



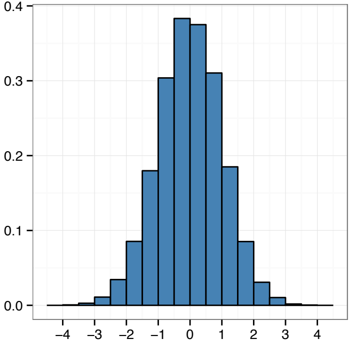


Figure: Empirical pdf (i.e., Scaled Histogram): Uniformity vs Nonuniformity (*from* *ZHANG Xiaowei*)

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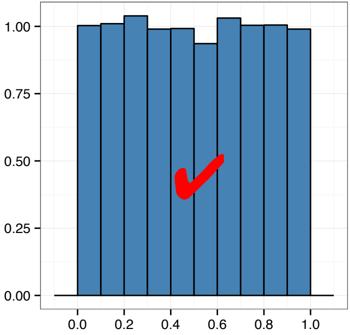

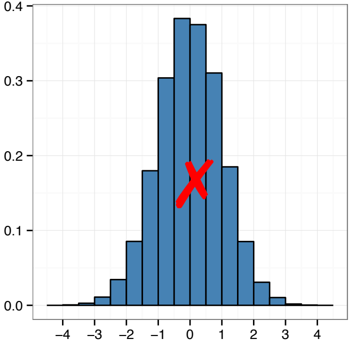

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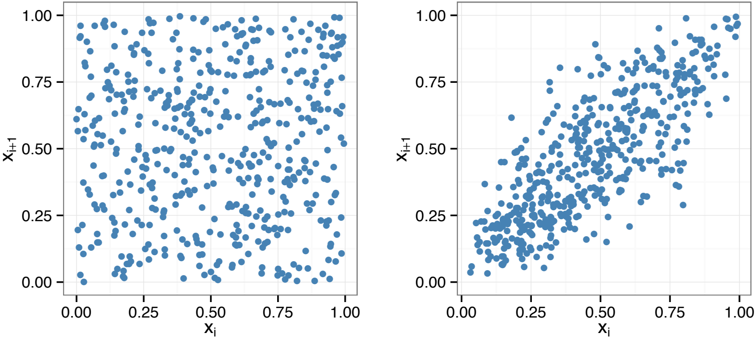

Figure: Scatter Plot: Uncorrelated vs Correlated (*from ZHANG Xiaowei*)

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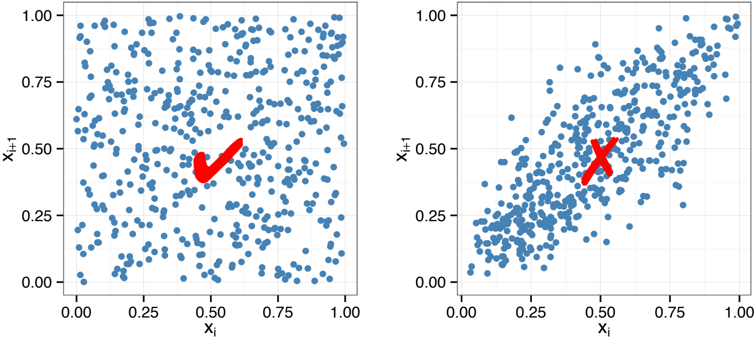

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  - Generating random numbers by a known method removes true randomness.
  - The set of pseudo-random numbers can be repeated.
- Goal: To produce a sequence of numbers in $[0, 1]$ that imitates the ideal properties of random numbers.
  - Statistical properties are the most important.
  - True randomness is not the first priority.

- Properties of a good random number generator (RNG):
  1. Pass statistical tests.
  2. Solid theoretical support.
  3. Fast.
  4. Sufficiently long cycle (period).
  5. Portable to different computers.
  6. Replicable.

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- Techniques for RNG:
  - Linear Congruential Generator (LCG)
  - Combined LCG
  - Multiple Recursive Generator (MRG)

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$$x_{i+1} = (ax_i + c) \bmod m, \quad i = 0, 1, 2, \ldots.$$

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- The selection of the values for a, c, m, and x_0 drastically affects the statistical properties and the cycle length.

- Example: Use LCG with $x_0 = 27$, $a = 17$, $c = 43$, and $m = 100$.

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\begin{aligned}
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- Try `https://xiaoweiz.shinyapps.io/randNumGen` for different parameters.

- An actual use of LCG (Lewis et al. 1969): $a = 7^5$, $c = 0$, $m = 2^{31} - 1 = 2{,}147{,}483{,}647$ (a prime number).
  - It adopts $u_i = \frac{x_i}{m+1}$.
  - It passes many of the standard statistical tests.
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- Note: By letting modulus m be a power of 2 (or close), the modulo operation can be conducted efficiently, since most digital computers use a binary representation of numbers.
- As computing power has increased, LCG is not adequate nowadays; more sophisticated RNGs are used in practice.

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1. Select seed $x_{1,0}$ in the range $[1, m_1 - 1]$ for the first generator, and seed $x_{2,0}$ in the range $[1, m_2 - 1]$ for the second. Set $j = 0$.
2. Calculate
$$x_{1,j+1} = a_1 x_{1,j} \bmod m_1,$$
$$x_{2,j+1} = a_2 x_{2,j} \bmod m_2.$$
3. Let $x_{j+1} = (x_{1,j+1} - x_{2,j+1}) \bmod (m_1 - 1)$.
(*Remark*: $\bmod$ uses floored division, i.e., $y \bmod m = y - m\lfloor \frac{y}{m} \rfloor$.)
4. Return
$$u_{j+1} = \begin{cases} \frac{x_{j+1}}{m_1}, & \text{if } x_{j+1} > 0, \\ \frac{m_1 - 1}{m_1}, & \text{if } x_{j+1} = 0. \end{cases}$$
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It has cycle length $(m_1 - 1)(m_2 - 1)/2 \approx 2 \times 10^{18}$.

- Multiple Recursive Generator (MRG): Extends LCG by using a higher-order recursion:

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- A specific instance that has been widely implemented is `MRG32k3a`[†] (L'Ecuyer 1999), which is a *combined MRG* with $J = 2$ and $K = 3$.

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- A specific instance that has been widely implemented is `MRG32k3a`† (L'Ecuyer 1999), which is a *combined MRG* with $J = 2$ and $K = 3$.
  - It has cycle length $\approx 3 \times 10^{57}$, which is enormous.
  - If you could generate one billion (10^9) pseudo-random numbers per second, then it would take longer than the age of the universe to exhaust the period of `MRG32k3a`!

† `MRG32k3a` or its adaptation is one of the RNGs used in MATLAB, R, SAS, Arena, etc.

- Tests based on generated sequences of numbers.
  - *Frequency Test* for uniformity (discussed in next lecture)
    – Kolmogorov–Smirnov test (柯尔莫哥洛夫–斯米尔诺夫检验)
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- Fortunately, the well-known RNGs which are widely used in simulation softwares and languages have been extensively tested and validated.
- Be careful when the RNG at hand is not explicitly known or documented!
  - Even RNGs that have been used for years in popular commercial softwares (e.g., Excel, Visual Basic), have been found to be inadequate (L'Ecuyer 2001).

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- Assumption: RNG is available, i.e. we have a sequence of random numbers (i.e., $\mathrm{Unif}(0,1)$ random variates).
- Goal: Produce random variates from a given probability distribution (e.g. exponential, Poisson, etc.).

Random Variate Generation

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- Goal: Produce random variates from a given probability distribution (e.g. exponential, Poisson, etc.).
- Widely-used techniques[†]
  - Inverse-transform technique (generic)
  - Acceptance-rejection technique (generic)
  - Other ad-hoc methods for some specific distributions

[†] Methods introduced in this lecture are exact; there are also approximation methods such as MCMC.

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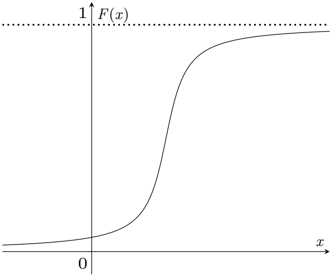


Figure: Continuous Random Variable

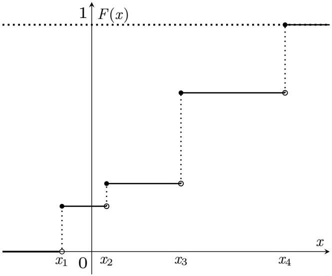


Figure: Discrete Random Variable

- Let $F(x)$ be the CDF of X, i.e., $F(x) = \mathbb{P}(X \leq x)$.

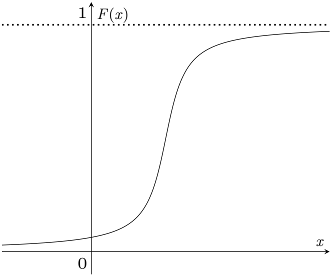
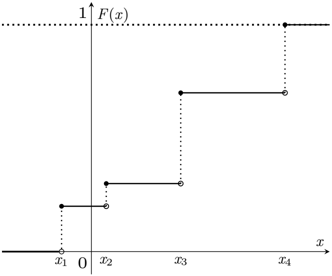


Figure: Continuous Random Variable



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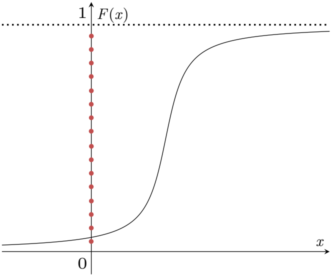

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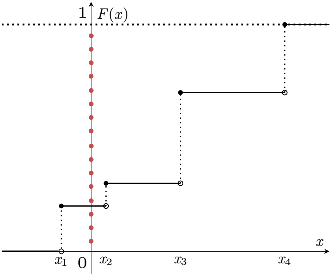

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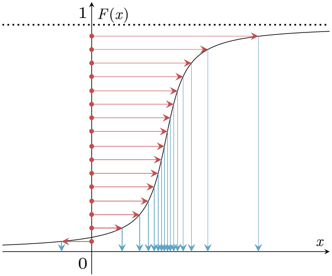

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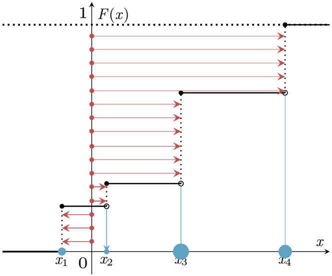

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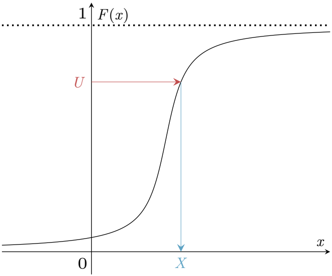
  1. Generate (as needed) random numbers (on vertical axis).
  2. Map inversely to points on horizontal axis, which are the desired random variates from $F(x)$.

- The formal definition of inverse function is

$$F^{-1}(y) := \min\{x : F(x) \geq y\}, \quad 0 < y < 1.$$



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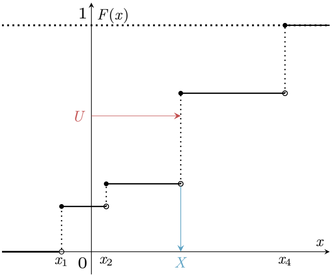

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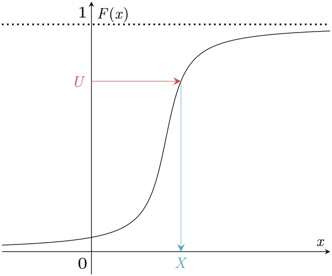

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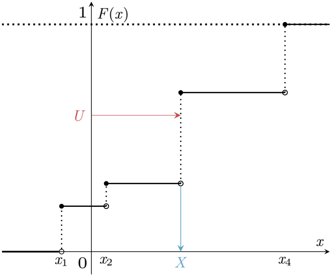

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- It can be used to sample from all (in principle) discrete distributions, e.g.,
    - discrete uniform
    - geometric
    - arbitrary empirical distribution

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$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise}, \end{cases} \quad F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b, \\ 1, & b < x. \end{cases}$$

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- *Remark*: $1 - U \sim \mathrm{Unif}(0,1) \Longrightarrow -\frac{1}{\lambda}\ln(U)$ is sufficient.
- Numerical test for $\mathrm{Exp}(1)$ in Excel.
  1. Generate 200 random numbers.
  2. Obtain 200 random variates via the inverse function.

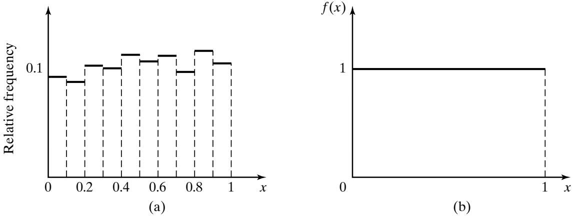

Figure:
(a) Empirical histogram of 200 generated uniform random numbers;
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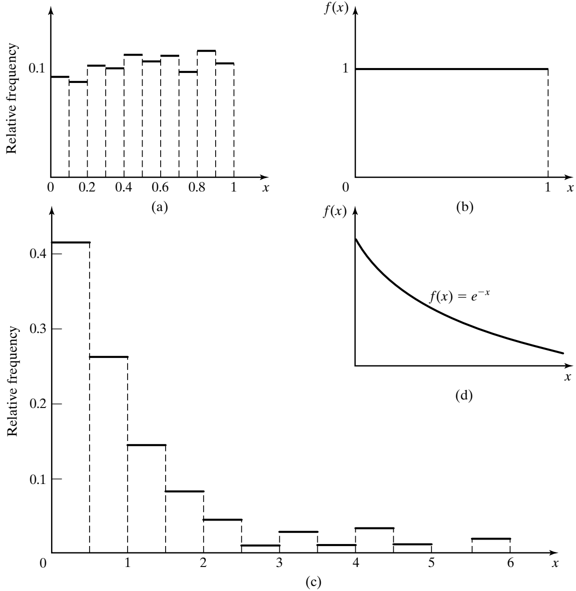

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(a) Empirical histogram of 200 generated uniform random numbers;
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(c) Empirical histogram of 200 generated variates from $\mathrm{Exp}(1)$;
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(*from* *Banks et al. (2010)*)

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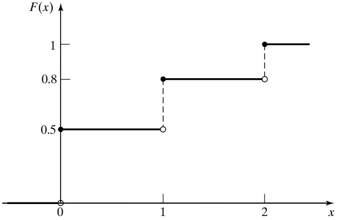

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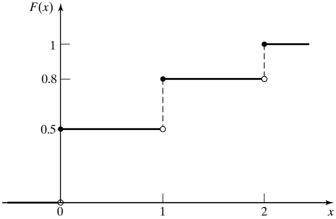

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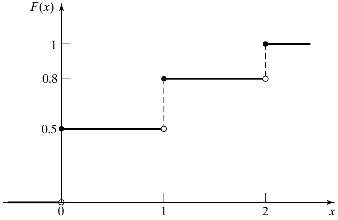

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Try it in Excel.

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- Acceptance-rejection technique is also useful for generating a *non-stationary Poisson process* (more details later).

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  - Whereas there exists a one-to-one mapping for the inverse-transform method.

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  - U itself does not have the desired distribution.
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- For $1/4 \leq x \leq 1$,

$$\mathbb{P}\{U \leq x | U \geq 1/4\} = \frac{\mathbb{P}\{U \leq x \text{ and } U \geq 1/4\}}{\mathbb{P}\{U \geq 1/4\}} = \frac{x - 1/4}{3/4},$$

which is exactly the desired CDF of $X \sim \mathrm{Unif}(1/4, 1)$.

- Suppose we want to generate random variates from X, whose density $f(x)$ has support $[a, b]$ and is upper bounded by M.

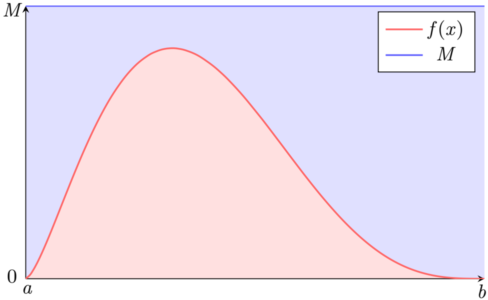

Figure: Bounded Support (*original image from ZHANG Xiaowei*)

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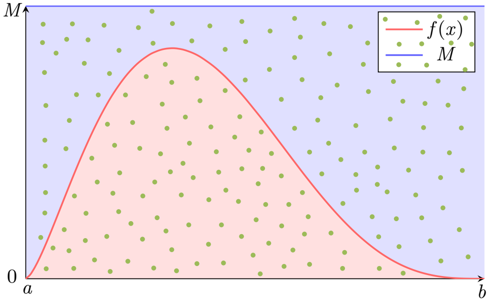

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1. Generate random variate pairs (y_1, z_1), (y_2, z_2), …, from $\text{uniform}\{(y, z) : a \le y \le b,\ 0 \le z \le M\}$.

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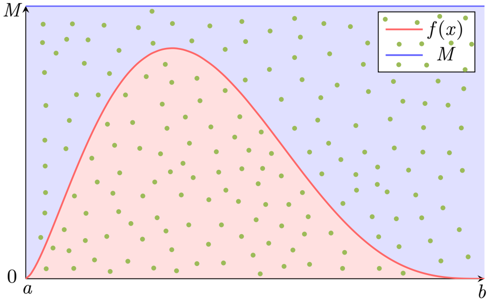

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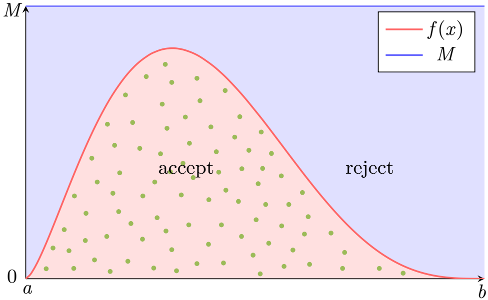

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2. Accept the pair if $z_i < f(y_i)$ and output y_i as random variate from X with density $f(x)$.

- Y conditioned on the event $\{Z < f(Y)\}$ has the same distribution as X, i.e., having density $f(x)$.
    - $(Y, Z) \sim \text{uniform}\{(y, z) : a \leq y \leq b,\ 0 \leq z \leq M\}$.

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$$\mathbb{P}\{Y \leq x | Z < f(Y)\}$$

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Note: $f_{Y,Z}(y, z) = \dfrac{1}{(b-a)M}$

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- Goal: Generate random variates from $\mathrm{Beta}(\alpha, \beta)$, where the density is $f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$, $x \in [0, 1]$.

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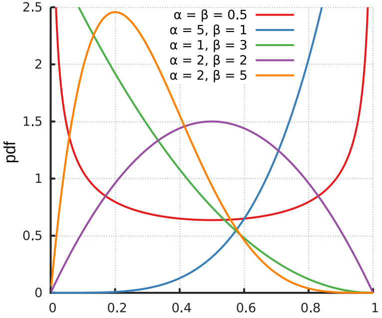

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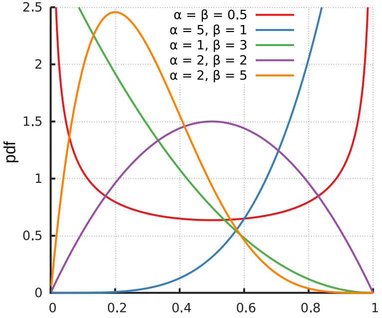

- If $\alpha > 1$ and $\beta > 1$, then $f(x)$ is maximized at $x = \frac{\alpha-1}{\alpha+\beta-2}$ and the maximum is $M = \frac{(\alpha-1)^{\alpha-1}(\beta-1)^{\beta-1}}{(\alpha+\beta-2)^{\alpha+\beta-2}B(\alpha,\beta)}$.

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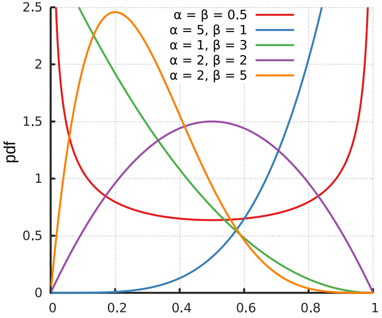

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- Generate random variates from X, whose density $f(x)$ is upper bounded by $Mg(x)$, where $g(x)$ is *instrumental* density.

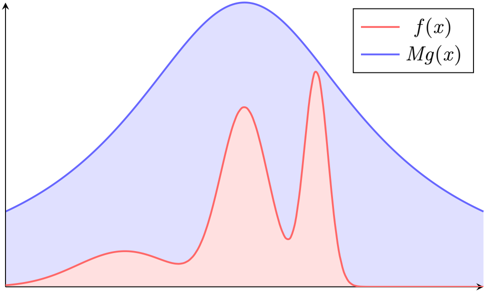

Figure: Unbounded Support (*original image from ZHANG Xiaowei*)

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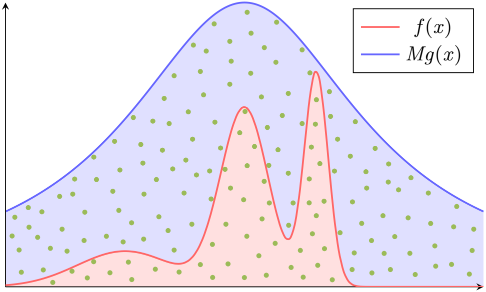

Figure: Unbounded Support (*original image from ZHANG Xiaowei*)

1. Generate random variate pairs (y_1, z_1), (y_2, z_2), ..., from $\text{uniform}\{(y, z) : y \in \text{support of } g(\cdot),\ 0 \le z \le Mg(y)\}$.

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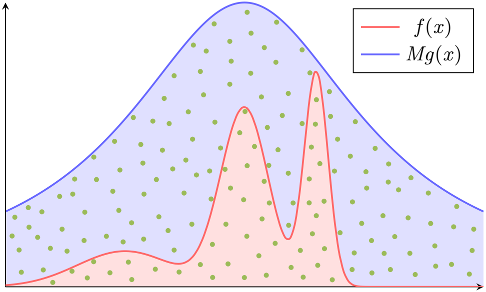

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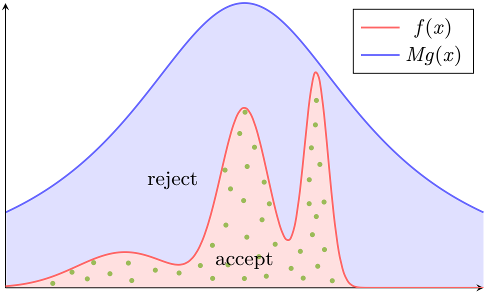

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2. Accept the pair if $z_i < f(y_i)$ and output y_i as random variate from X with density $f(x)$.

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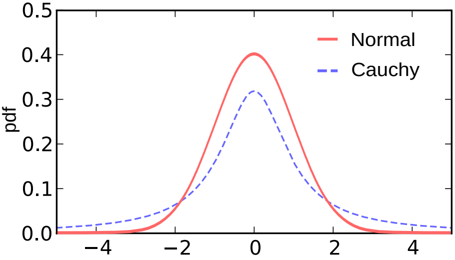

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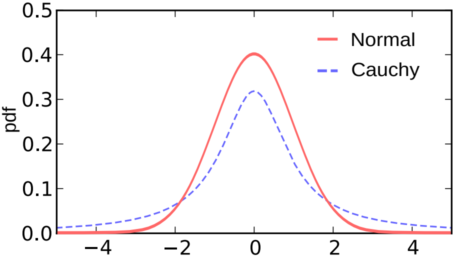
- Use $\mathrm{Cauchy}(0,1)$ density as instrumental density, which is $g(x) = \frac{1}{\pi(1+x^2)}$, $x \in (-\infty, \infty)$.



- It is easy to see that $\frac{f(x)}{g(x)} = \sqrt{\frac{\pi}{2}}(1+x^2)e^{-\frac{x^2}{2}}$ is maximized at $x = \pm 1$ and the maximum is $\sqrt{\frac{2\pi}{e}}$, which is the required M.

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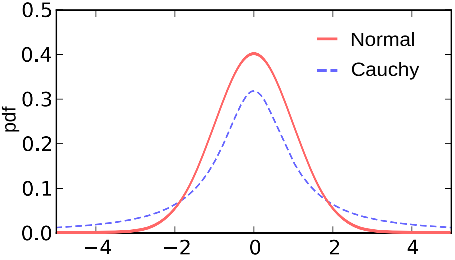

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- Box–Muller method for $\mathcal{N}(0,1)$ random variates:
  1. Generate u_1 and u_2 independently from $\mathrm{Unif}(0,1)$.
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- Intuition:
  - For two independent $\mathcal{N}(0,1)$ RVs Z_1 and Z_2,
  $$Z_1^2, Z_2^2 \sim \chi_1^2,\ Z_1^2 + Z_2^2 \sim \chi_2^2.$$
  - $X \sim \mathrm{Exp}(1/2) \iff X \sim \chi_2^2$.
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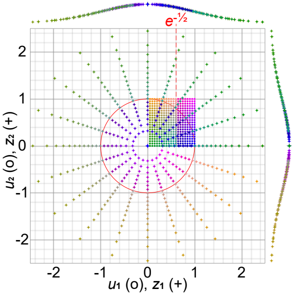

Figure: Box–Muller Method Visualisation
(image by Cmglee / CC BY 3.0)
Interactive Graph

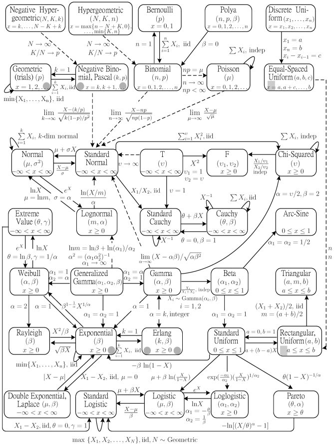

Figure: Relationships Among 35 Distributions (from Song (2005))

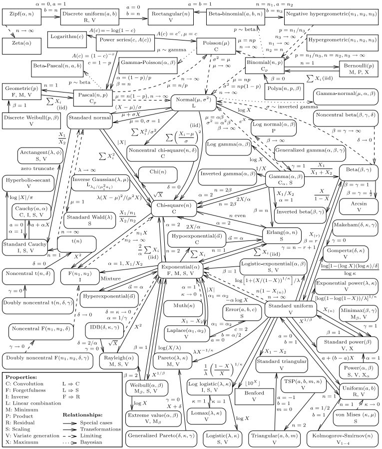

Figure: Relationships Among 76 Distributions (from Leemis & McQueston (2008))

- **Poisson process** with rate λ: Interarrival time distribution is exponential with rate λ (or mean $1/\lambda$), and

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- **Nonhomogeneous Poisson process** with rate (intensity) function $\lambda(t)$:

$$N(t+h) - N(t) \sim \text{Poisson}(m(t+h) - m(t)),$$

where $m(t) = \int_0^t \lambda(s)\mathrm{d}s$.

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  4. Go to Step 2.